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SciVerse ScienceDirect
Indagationes Mathematicae 21 (2011) 138–155
indagationes mathematicae
www.elsevier.com/locate/indag

The order convergence structure

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Received 20 September 2010; received in revised form 24 January 2011; accepted 9 February 2011

Communicated by Dr. B. de Pagter

Abstract

In this paper, we study order convergence and the order convergence structure in the context of σ-distributive lattices. Particular emphasis is placed on spaces with additional algebraic structure: we show that on a Riesz algebra with σ-order continuous multiplication, the order convergence structure is an algebra convergence structure, and construct the convergence vector space completion of an Archimedean Riesz space with respect to the order convergence structure.

MSC: 06A19; 06F25; 54C30

Keywords: Order convergence; Convergence structure; Riesz space

1. Introduction

A useful notion of convergence of sequences on a poset L is that of order convergence; see for instance [2,7]. Recall that a sequence (u_n) on L order converges to $u \in L$ whenever there is an increasing sequence (λ_n) and a decreasing sequence (μ_n) on L such that

$$\sup_{n\in\mathbb{N}} \lambda_n = u = \inf_{n\in\mathbb{N}} \mu_n \quad \text{and} \quad \lambda_n \leq u_n \leq \mu_n, \quad n \in \mathbb{N}. \tag{1}$$

In case the poset L is a Riesz space, the relation (1) is equivalent to the following: there exists a sequence (λ_n) that decreases to 0 such that

$$|u - u_n| \leq \lambda_n, \quad n \in \mathbb{N}. \tag{2}$$

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0019-3577/$ - see front matter
doi:10.1016/j.indag.2011.02.004

In general, there is no topology on a poset L that induced the convergence (1) on L; see for instance [2] for an example. Thus, while order convergence is clearly a topological phenomenon, it cannot be described in terms of usual topology. Therefore a more general framework is required.

The notion of a convergence structure [4] is a useful generalization of that of a topology. Several authors have studied order convergence, and related concepts, on certain classes of posets in the context of convergence structures. Here we may recall Ball's work [3] on group convergence structures on lattice ordered groups, and Papangelou's construction [8] of the sequential convergence group completion of commutative lattice ordered groups with respect to order convergence. Recently [2] a vector space convergence structure, called the *order convergence structure*, inducing order convergence of sequences was defined on arbitrary Archimedean Riesz spaces. The main objective of [2] is the construction of a completion of $\mathcal{C}(X)$ with respect to the order convergence structure.

In this paper we consider the order convergence structure in the more general context of σ-distributive lattices. In particular, we show that for any such lattice L, the order convergence structure is indeed a convergence structure, and that it induces order convergence of sequences. Countability and separation properties of the order convergence structure are studied. As a particular case, we consider the order convergence structure on an Archimedean Riesz space E. The convergence vector space completion of E with respect to the order convergence structure is constructed and compared with the Dedekind σ-completion of van Haandel and van Rooij [14]. It is also shown that if E is an Archimedean Riesz algebra, then the order convergence structure is an algebra convergence structure if and only if the multiplication in E is σ-order continuous, and the convergence vector space completion of E is a convergence algebra. The continuity of linear operators from a Riesz space E into another Riesz space F, with respect to the order convergence structure, is characterized in terms of the class of σ-order continuous operators.

For details on convergence structures, and the order convergence structure in particular, we refer the reader to [2,4].

2. Convergence structures

We recall the definition of a convergence structure and related concepts. All definitions are taken from [4], and we refer the reader to this text for more details on convergence structures and their applications.

Definition 1. A convergence structure on a set X is a mapping $\lambda : X \to \mathcal{P}(\mathcal{F}(X))$, where $\mathcal{F}(X)$ denotes the set of proper filters on X, and $\mathcal{P}(\mathcal{F}(X))$ its powerset, that satisfies the following conditions:

(i) $[x] \in \lambda(x)$ for every $x \in X$.
(ii) If $\mathcal{F} \in \lambda(x)$ and $\mathcal{G} \supseteq \mathcal{F}$, then $\mathcal{G} \in \lambda(x)$.
(iii) If $\mathcal{F} \in \lambda(x)$ and $\mathcal{G} \in \lambda(x)$, then $\mathcal{F} \cap \mathcal{G} \in \lambda(x)$.

Remark 2. The following notation will be used throughout the paper. If $\mathcal{B}$ is a collection of nonempty subsets of X which is downward directed with respect to inclusion, then $[\mathcal{B}] = \{F \subseteq X : B \subseteq F \text{ for some } B \in \mathcal{B}\}$ is the *filter generated by* $\mathcal{B}$, and $\mathcal{B}$ is a *basis* the filter $[\mathcal{B}]$. In particular, if $\mathcal{B}$ contains only the singleton $\{x\}$, then we denote $[\mathcal{B}]$ by $[x]$. This is the principle ultrafilter generated by x.

A sequence (x_n) on a convergence space X converges to $x \in X$ if and only if the Frechét filter associated with (x_n), that is, the filter

$$\langle x_n \rangle = [\{\{x_n : n \geq k\} : k \in \mathbb{N}\}]$$

converges to x.

A convergence structure λ on a set X is *first countable* if for all $x \in X$ and $\mathcal{F} \in \lambda(x)$ there exists $\mathcal{G} \in \lambda(x)$ with a countable basis such that $\mathcal{G} \subseteq \mathcal{F}$. The convergence structure λ is called *Hausdorff* whenever $\lambda(x) \cap \lambda(y) = \emptyset$ for all $x \neq y$ in X, while λ is *regular* if $a_\lambda(\mathcal{F}) \in \lambda(x)$ whenever $\mathcal{F} \in \lambda(x)$. Here a_λ denotes the *adherence operator* with respect to λ. That is, for $F \subseteq X$ we have $a_\lambda(F) = \{x \in X : F \in \mathcal{F} \text{ for some } \mathcal{F} \in \lambda(x)\}$. The filter $a_\lambda(\mathcal{F})$ is the filter generated by the collection $\{a_\lambda(F) : F \in \mathcal{F}\}$. A set $F \subseteq X$ is *closed* if $a_\lambda(F) = F$.

Let X and Y be convergence spaces. A mapping $f : X \to Y$ is *continuous* if, for every $x \in X$ and every $\mathcal{F} \in \lambda(x)$, the filter $f(\mathcal{F}) = [\{f(F) : F \in \mathcal{F}\}]$ converges to $f(x)$ in Y. A filter $\mathcal{H}$ on $X \times Y$ converges to $(x, y) \in X \times Y$ with respect to the *product convergence structure* if and only if

$$\mathcal{H} \supseteq \mathcal{F} \times \mathcal{G} = [\{F \times G : F \in \mathcal{F},\ G \in \mathcal{G}\}]$$

where $\mathcal{F}$ converges to x in X, while $\mathcal{G}$ converges to y in Y.

Let X be a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A convergence structure on X is a *vector space convergence structure*, and X a *convergence vector space*, if both the mappings $+ : X \times X \ni (x, y) \mapsto x + y \in X$ and $\cdot : \mathbb{K} \times X \ni (\alpha, x) \mapsto \alpha x \in X$ are continuous. Here $X \times X$ and $\mathbb{K} \times X$ are equipped with the product convergence structure. In particular, if X is an algebra, then a convergence structure on X is an *algebra convergence structure* whenever it is a vector space convergence structure, and the bilinear mapping $\cdot : X \times X \ni (x, y) \mapsto xy \in X$ is continuous.

A subset A of a convergence vector space is called *bounded* if the filter $[\{\{\epsilon x : x \in A, |\epsilon| < \frac{1}{n}\} : n \in \mathbb{N}\}]$ converges to 0 in X. If Y is another convergence vector space, then a linear mapping $T : X \to Y$ is called *bounded* if $T(A)$ is bounded in Y for every bounded subset A of X. Every continuous linear mapping is bounded.

A filter $\mathcal{F}$ on a convergence vector space is called a *Cauchy filter* if the filter $\mathcal{F} - \mathcal{F} = [\{\{x - y : x, y \in F\} : F \in \mathcal{F}\}]$ converges to 0 in X. In particular, a sequence (x_n) in X is a *Cauchy sequence* if the filter $\langle x_n \rangle - \langle x_n \rangle$ converges to 0. Every convergent filter on X is a Cauchy filter. If every Cauchy filter on X converges to some $x \in X$, then X is called *complete*. Now suppose that X is Hausdorff. A Hausdorff convergence vector space $X^\sharp$ is called a *completion of* X if there exists a continuous, injective linear mapping $\iota_X : X \to X^\sharp$, with a continuous inverse defined on $\iota_X(X)$, that satisfies the following *universal property*: for every complete Hausdorff convergence vector space Y, and each continuous linear mapping $T : X \to Y$ there exists a unique continuous linear mapping $T^\sharp : X^\sharp \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "\iota_X"'] & & Y \\ & X^\sharp \arrow[ur, "\exists ! T^\sharp"'] & \end{tikzcd} \tag{3}$$

If $X^\sharp$ is a completion of X, then $\iota_X(X)$ is *dense* in $X^\sharp$. That is, for all $x \in X^\sharp$ there exists a filter $\mathcal{F}$ on $X^\sharp$ such that $\mathcal{F}$ converges to x and

$$F \cap \iota_X(X) \neq \emptyset, \quad F \in \mathcal{F}. \tag{4}$$

Any convergence vector space can have at most one completion, up to convergence vector space isomorphism. However, not every Hausdorff convergence vector space has a completion; see [6].

3. Order convergence structure

As mentioned in Section 1, order convergence of sequences on a poset L is, in general, not induced by a topology. However, as is shown in [2], if L is a lattice, and order convergence of sequences on L is induced by a convergence structure, then the relation

$$\mathcal{F} \in \lambda_o(u) \Leftrightarrow \{[\lambda_n, \mu_n] : n \in \mathbb{N}\} \subseteq \mathcal{F}, \tag{5}$$

with (λ_n), $(\mu_n) \subset L$ increasing and decreasing to u, respectively, defines a convergence structure on L which induces order convergence of sequences. That is, if L is an $\mathcal{FS}$-space [2] with respect to order convergence of sequences, then (5) is a convergence structure which induces order convergence of sequences. In particular, this is the case when L is a Riesz space.

3.1. σ-distributive lattices

We now proceed to generalize the results obtained in [2] to the class of σ-distributive lattices.

Definition 3. A lattice L is called σ-distributive if the distributive laws

$$v \vee a_0 = \inf\{v \vee a : a \in A\}, \qquad v \wedge b_0 = \sup\{v \wedge b : b \in B\}$$

hold for all $v \in L$ and countable sets $A, B \subseteq L$ with $a_0 = \inf A$ and $b_0 = \sup B$.

Theorem 4. *If L is a σ-distributive lattice, then (5) defines a convergence structure on L.*

Proof. Axioms (i) and (ii) of Definition 1 are trivially satisfied, so it remains to verify (iii). In this regard, assume $\mathcal{F} \in \lambda_o(u)$ and $\mathcal{G} \in \lambda_o(u)$ for some $u \in L$. Let (λ_n) and (μ_n) be the increasing, respectively decreasing, sequences associated with $\mathcal{F}$ through (5), while (λ'_n) and (μ'_n) are the sequences associated with $\mathcal{G}$. Since (λ_n) and (λ'_n) both increase to u, it follows by Lemma 5 that the sequence $(\hat{\lambda}_n)$ defined through $\hat{\lambda}_n = \lambda_n \wedge \lambda'_n,\ n \in \mathbb{N}$ increases to u. Similarly, the sequence $(\hat{\mu}_n)$ defined as $\hat{\mu}_n = \mu_n \vee \mu'_n,\ n \in \mathbb{N}$ decreases to u. Furthermore, it is clear from (5) that $\hat{\lambda}_n \le \lambda_n \le \mu_n \le \hat{\mu}_n, \hat{\lambda}_n \le \lambda'_n \le \mu'_n \le \hat{\mu}_n$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we have $[\lambda_n, \mu_n] \cup [\lambda'_n, \mu'_n] \subseteq [\hat{\lambda}_n, \hat{\mu}_n],\ n \in \mathbb{N}$ so that $[\{[\hat{\lambda}_n, \hat{\mu}_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F} \cap \mathcal{G}$. Since $(\hat{\lambda}_n)$ increases to u and $(\hat{\mu}_n)$ decreases to u, it follows that $\mathcal{F} \cap \mathcal{G} \in \lambda_o(u)$. This completes the proof. □

The proof of Theorem 4 depends on the following.

Lemma 5. *Let L be a σ-distributive lattice. For sequences (u_n) and (v_n) in L, the following holds.*

(i) *If (u_n) increases to u and (v_n) increases to v, then the sequence $(u_n \wedge v_n)$ increases to $u \wedge v$.*
(ii) *If (u_n) decreases to u and (v_n) decreases to v, then the sequence $(u_n \vee v_n)$ decreases to $u \vee v$.*

Proof. We only prove (i), since (ii) follows in the same way. For every $m,\ n \in \mathbb{N}$ it follows from

$$u_n \leq u_{n+1}, \qquad v_m \leq v_{m+1} \tag{6}$$

that $u_n \wedge v_m \leq u_n \wedge v_{m+1}$. Therefore the sequence $(w_m^{(n)}) = (u_n \wedge v_m)$ is increasing for every $n \in \mathbb{N}$. Furthermore, since L is σ-distributive, it follows from $v = \sup_{m\in\mathbb{N}} v_m$ that $u_n \wedge v = \sup_{m\in\mathbb{N}} w_m^{(n)},\ n \in \mathbb{N}$. But (6) implies that $u_n \wedge v \leq u_{n+1} \wedge v$ for all $n \in \mathbb{N}$. Therefore the sequence $(u_n \wedge v)$ is increasing, and since L is σ-distributive, it follows form $\sup_{n\in\mathbb{N}} u_n = u$ that $(u_n \wedge v)$ increases to $u \wedge v$. According to [2, Lemma 36], the sequence (x_n) defined through $x_n = \sup\{w_1^{(n)}, \ldots, w_n^{(n)}\}$ increases to $(u \wedge v)$. But (6) implies that $x_n = \sup\{u_n \wedge v_1, \ldots, u_n \wedge v_n\} = u_n \wedge v_n$. This completes the proof. $\square$

3.2. Convergence of sequences

This section is concerned with the characterization of the convergent sequences with respect to λ_o. In this regard we have the following.

Theorem 6. *Suppose that L is a lattice such that λ_o is a convergence structure on L. A sequence (u_n) in L converges to $u \in L$ with respect to λ_o if and only if (u_n) order converges to u.*

Proof. Let (u_n) be a sequence on L, and assume that (u_n) converges to u with respect to λ_o. Let (λ_n) and (μ_n) be the sequences associated with the filter $\langle u_n \rangle$ through (5), so that $[\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}] \subseteq \langle u_n \rangle$. This inclusion implies that for each $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $\lambda_n \leq u_k \leq \mu_n$ for all $k \geq m_n$. Note that we may select the sequence (m_n) is such a way that it is strictly increasing. Now define the sequences (λ'_n) and (μ'_n) as follows:

$$\lambda'_n = \begin{cases} \inf\{\lambda_1, u_1, \ldots, u_{m_1}\} & \text{if } n < m_1 \\ \lambda_{m_n} & \text{if } m_n \leq n < m_{n+1},\ n > 2 \end{cases}$$

$$\mu'_n = \begin{cases} \sup\{\mu_1, u_1, \ldots, u_{m_1}\} & \text{if } n < m_1 \\ \mu_{m_n} & \text{if } m_n \leq n < m_{n+1},\ n > 2. \end{cases}$$

Since the sequence (m_n) of integers is increasing, and the sequences (λ_n) and (μ_n) are increasing and decreasing, respectively, it follows that (λ'_n) is increasing and (μ'_n) is decreasing. Furthermore, from the respective definitions of the sequences and (λ'_n) and (μ'_n) it follows that $\lambda'_n \leq \lambda'_{m_n} \leq \mu'_{m_n} \leq \mu_n$ for all $n \in \mathbb{N}$. Therefore $\sup_{n\in\mathbb{N}} \lambda'_n = \sup_{n\in\mathbb{N}} \lambda_n = u = \inf_{n\in\mathbb{N}} \mu_n = \inf_{n\in\mathbb{N}} \mu'_n$. Since $\lambda_n \leq u_k \leq \mu_n$ for all $k \geq m_n$, it follows that $\lambda'_n \leq u_n \leq \mu'_n,\ n \in \mathbb{N}$. Therefore the sequences (u_n) order converges to u in L.

The converse is trivial. $\square$

Theorem 6 is the converse of [2, Theorem 16(iii)]. In particular, we see that if L is a lattice such that λ_o is a convergence structure on L, then order convergence of sequences on L is induced by a convergence structure. Combining this observation with [2, Theorem 16(iii)] we obtain the following characterization of those lattices on which order convergence of sequences is induced by a convergence structure.

Corollary 7. *Let L be a lattice. Then order convergence of sequences is induced by a convergence structure if and only if λ_o is a convergence structure on L.*

A particular case of Corollary 7 is the following.

Corollary 8. *If L is a σ-distributive lattice, then order convergence of sequences is induced by the order convergence structure.*

The order convergence structure is not the unique convergence structure that induces order convergence of sequences. Indeed, if order convergence of sequences is induced by a convergence structure, then the relation

$$\mathcal{F} \in \lambda(u) \Leftrightarrow \begin{pmatrix} \forall & (u_n) \subseteq L, \langle u_n \rangle \subseteq \mathcal{F} : \\ & (u_n) \text{ order converges to } u \end{pmatrix} \tag{7}$$

defines the finest convergence structure on L that induces order convergence of sequences. However, it should be noted that, unlike the order convergence structure, the convergence structure (7) will in general fail to be compatible with additional algebraic structure on the lattice L. In particular, if L is an Archimedean vector lattice, then λ_o is a vector space convergence structure, while (7) is not.

On the other hand,

$$\mathcal{F} \in \lambda(u) \Leftrightarrow \begin{pmatrix} \forall & (u_n) \subseteq L, \langle u_n \rangle \supseteq \mathcal{F} : \\ & (u_n) \text{ order converges to } u \end{pmatrix} \tag{8}$$

defines the coarsest convergence structure on L inducing order convergence of sequences. However, while order convergence of sequences in a poset L is a generalization of convergence of sequences of real numbers, the convergence structure (8) does not generalize the usual metric topology of $\mathbb{R}$. Indeed, see [4, page 56], in case $L = \mathbb{R}$, the convergence structure (8) is strictly coarser than the customary topological convergence structure on $\mathbb{R}$. However, as can easily be seen, if L is the field $\mathbb{R}$, the order convergence structure λ_o agrees with the usual metric topology. Thus λ_o is a natural generalization of the usual metric topology on $\mathbb{R}$ to all σ-distributive lattices.

3.3. Properties of the order convergence structure

We now establish some basic properties of the convergence structure λ_o. In particular, we establish the countability and separation properties of this convergence structure. As an immediate corollary to Theorem 4, we see that λ_o is first countable.

Corollary 9. *Let L be a σ-distributive lattice. Then λ_o is a first countable convergence structure on L.*

The order convergence structure satisfies the following separation axioms.

Proposition 10. *Let L be a σ-distributive lattice. Then λ_o is a regular convergence structure on L.*

Proof. Since λ_o is first countable by Corollary 9, it follows [4, Proposition 1.6.4] that a set $A \subseteq L$ is closed if and only if A is sequentially closed. Consider a nonempty order interval $I = [\lambda, \mu] \subset L$, and a sequence $(u_n) \subset I$ that converges to $u \in L$ with respect to λ_o. Theorem 6 implies that (u_n) order converges to u. Let (λ_n) and (μ_n) be the increasing, respectively decreasing, sequences associated with (u_n) through (1). Since $\lambda_n \leq u_n \leq \mu_n$ and $\lambda \leq u_n \leq \mu$ for all $n \in \mathbb{N}$, it follows that

$$\lambda_n \leq \hat{\lambda}_n = \lambda_n \vee \lambda \leq u_n \leq \mu \wedge \mu_n = \hat{\mu}_n \leq \mu_n, \quad n \in \mathbb{N}. \tag{9}$$

Since the sequence (λ_n) increases to u, and L is σ-distributive, it follows that the sequence $(\hat{\lambda}_n)$ increases to $u \vee \lambda$. Similarly, the sequence $(\hat{\mu}_n)$ decreases to $u \wedge \mu$. But (9) implies that $\sup_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \hat{\lambda}_n \leq \inf_{n\in\mathbb{N}} \hat{\mu}_n \leq \inf_{n\in\mathbb{N}} \mu_n$. Since (λ_n) increases to u and (μ_n) decreases to u, it now follows that $u \vee \lambda = \sup_{n\in\mathbb{N}} \hat{\lambda}_n = u = \inf_{n\in\mathbb{N}} \hat{\mu}_n = u \wedge \mu$. Therefore $\lambda \leq u \leq \mu$ so that the order interval $[\lambda, \mu]$ is sequentially closed with respect to λ_o, and hence closed with respect to λ_o.

The definition of the order convergence structure given in (5) now implies that every filter $\mathcal{F}$ that converges with respect to λ_o contains a filter with a basis of closed sets that converges to the same limit. From this it follows that λ_o is regular. □

Corollary 11. *If L is a σ-distributive lattice, then λ_o is a Hausdorff convergence structure.*

Proof. Singletons are clearly closed with respect to λ_o. Thus the result follows from Proposition 10; see [4, Section 1.4]. □

4. Order convergence in Riesz spaces

In this section we consider a particular case of the results presented in Section 3. We recall [7, Theorem 12.2] that any Riesz space F is a σ-distributive lattice. Therefore an immediate consequence of Theorems 4 and 6, Proposition 10 and Corollaries 8 and 9 is the following.

Corollary 12. *If F is a Riesz space, then (5) defines a Hausdorff, regular and first countable convergence structure on F. Furthermore, a sequence (u_n) on F converges to $u \in F$ if and only if (u_n) order converges to u in F.*

Remark 13. Unless otherwise indicated, convergence of filters on a Riesz space F will refer to convergence with respect to the order convergence structure. In particular, a sequence on a Riesz space converges to some $u \in F$ if and only if it order converges to u.

Corollary 12 is partly known. Furthermore, the following also holds; see [2, Theorem 17].

Theorem 14. *If F is an Archimedean Riesz space, then the convergence structure λ_o is a vector space convergence structure on L.*

The following result relates the lattice structure of F to the convergence structure λ_o.

Proposition 15. *Let F be an Archimedean Riesz space. Then the bilinear mappings $\bigvee : F \times F \ni (u, v) \mapsto u \vee v \in F$ and $\bigwedge : F \times F \ni (u, v) \mapsto u \wedge v \in F$ are continuous.*

Proof. Consider filters $\mathcal{F}$ and $\mathcal{G}$ that converge to u and v, respectively. Let (λ_n) and (μ_n) be the increasing and decreasing sequences associated with $\mathcal{F}$ through (5). Similarly, (λ'_n) and (μ'_n) are the sequences associated with $\mathcal{G}$ through (5). According to [7, Theorem 15.3] the sequences $(\hat{\lambda}_n)$ and $(\hat{\mu}_n)$, defined as $\hat{\lambda}_n = \lambda_n \vee \lambda'_n$ and $\hat{\mu}_n = \mu_n \vee \mu'_n$, increases and decreases to $u \vee v$, respectively, so that $\mathcal{H} = [\{[\hat{\lambda}_n, \hat{\mu}_n] : n \in \mathbb{N}\}]$ converges to $u \vee v$. But $\hat{\lambda}_n \leq w \vee w' \leq \hat{\mu}_n$ for all $n \in \mathbb{N}$, $w \in [\lambda_n, \mu_n]$ and $w' \in [\lambda'_n, \mu'_n]$. Since $[\lambda_n, \mu_n] \in \mathcal{F}$ and $[\lambda'_n, \mu'_n] \in \mathcal{G}$ according to (5), it follows that $\mathcal{H} \subseteq \bigvee(\mathcal{F}, \mathcal{G})$, where $\bigvee(\mathcal{F}, \mathcal{G}) = [\{\{u \vee v : u \in F,\ v \in G\} : F \in \mathcal{F}, G \in \mathcal{G}\}]$ is the image of the filter $\mathcal{F} \times \mathcal{G}$ under the mapping $\bigvee$. Therefore $\vee(\mathcal{F}, \mathcal{G})$ converges to $u \vee v$, since $\mathcal{H}$ converges to $u \vee v$. This verifies that the mapping $\bigvee$ is continuous. The continuity of the mapping $\bigwedge$ follows in the same way. □

4.1. Convergence on Riesz algebras

We now consider the case when the Riesz space F is also an algebra. We will show that the order convergence structure is an algebra convergence structure, provided that F is a Riesz algebra with σ-order continuous multiplication. We recall [5,15] that a *Riesz algebra* is a Riesz space F that is also an associative algebra such that

$$F^+ \cdot F^+ \subseteq F^+. \tag{10}$$

Here F^+ denotes the positive cone of F, given by $F^+ = \{f \in F : f \geq 0\}$. Note that the inclusion (10) is equivalent to the inequality

$$f \leq g \Rightarrow fh \leq gh, \tag{11}$$

where $f, g \in F$ and $h \in F^+$. The multiplication in a Riesz algebra is called *σ-order continuous* if

$$\sup\{ab : a \in A, b \in B\} = a_0 b_0 \tag{12}$$

holds for all countable sets $A, B \subseteq F^+$ such that $a_0 = \sup A$ and $b_0 = \sup B$.

In any Riesz Algebra F, the identity

$$fg = f^+g^+ + f^-g^- - f^+g^- - f^-g^+ \tag{13}$$

holds for all $f, g \in F$. Here f^+, f^- and $|f|$ denote the positive part and negative part of f, respectively, given by $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

Theorem 16. *Let F be an Archimedean Riesz algebra. Then the order convergence structure is an algebra convergence structure on F if and only if F the multiplication in F is σ-order continuous.*

Proof. Assume that F has σ-order continuous multiplication. In view of Theorem 14, it is sufficient to prove the continuity of multiplication. In this regard, let $\mathcal{F}$ and $\mathcal{G}$ be filters on F that converge to f and g, respectively. Without loss of generality, we may assume that $\mathcal{F}$ and $\mathcal{G}$ are of the form $\mathcal{F} = [\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}]$ and $\mathcal{G} = [\{[\hat{\lambda}_n, \hat{\mu}_n] : n \in \mathbb{N}\}]$, where (λ_n) and $(\hat{\lambda}_n)$ increase to f and g, while (μ_n) and $(\hat{\mu}_n)$ decrease to f and g, respectively. Let $\mathcal{F}^+$ and $\mathcal{F}^-$ denote the filters $\mathcal{F}^+ = [\{\{u^+ : \lambda_n \leq u \leq \mu_n\} : n \in \mathbb{N}\}]$ and $\mathcal{F}^- = [\{\{u^- : \lambda_n \leq u \leq \mu_n\} : n \in \mathbb{N}\}]$. The filters $\mathcal{G}^+$ and $\mathcal{G}^-$ are similarly defined. Note that $\mathcal{F}^+$ is the image of the filter $[0] \times \mathcal{F}$ under the mapping $\bigvee$. Therefore it follows from Proposition 15 that $\mathcal{F}^+$ converges to f^+. Similarly, $\mathcal{F}^-$ converges to f^- while $\mathcal{G}^+$ converges to g^+ and $\mathcal{G}^-$ converges to g^-.

The inclusions $\mathcal{F}^+ - \mathcal{F}^- \subseteq \mathcal{F}, \mathcal{G}^+ - \mathcal{G}^- \subseteq \mathcal{G}$ are direct consequences of the decomposition $u = u^+ - u^-$ of an element of F into its positive and negative parts. These inclusions imply that

$$\mathcal{H} = (\mathcal{F}^+ - \mathcal{F}^-) \cdot (\mathcal{G}^+ - \mathcal{G}^-) \subseteq \mathcal{F} \cdot \mathcal{G}. \tag{14}$$

The filter $\mathcal{H}$ is based on sets of the form

$$H = \{(u_0^+ - v_0^-)(u_1^+ - v_1^-) : u_0, v_0 \in F, u_1, v_1 \in G\}$$

where $F \in \mathcal{F}$ and $G \in \mathcal{G}$. A standard computation shows that

$$\begin{aligned} H &= \{u_0^+u_1^+ - u_0^+v_1^- - v_0^-u_1^+ + v_0^-v_1^- : u_0, v_0 \in F, u_1, v_1 \in G\} \\ &\subseteq \{u_0^+u_1^+ : u_0 \in F, u_1 \in G\} - \{u_0^+v_1^- : u_0 \in F, v_1 \in G\} \end{aligned}$$

$$-\{v_0^- u_1^+ : v_0 \in F, u_1 \in G\} + \{v_0^- v_1^- : v_0 \in F, v_1 \in G\}. \tag{15}$$

The inclusions (14) and (15) show that

$$\mathcal{F}^+ \cdot \mathcal{G}^+ + \mathcal{F}^- \cdot \mathcal{G}^- - \mathcal{F}^+ \cdot \mathcal{G}^- - \mathcal{F}^- \cdot \mathcal{G}^+ \subseteq \mathcal{F} \cdot \mathcal{G}. \tag{16}$$

Since $\lambda_n^+ \leq u^+ \leq \mu_n^+, \mu_n^- \leq u^- \leq \lambda_n^-$ for all $n \in \mathbb{N}$ and $u \in [\lambda_n, \mu_n]$, it follows that $\{u^+ : \lambda_n \leq u \leq \mu_n\} \subseteq [\lambda_n^+, \mu_n^+]$ and $\{u^- : \lambda_n \leq u \leq \mu_n\} \subseteq [\mu_n^-, \lambda_n^-]$ so that $\{[\lambda_n^+, \mu_n^+] : n \in \mathbb{N}\} \subseteq \mathcal{F}^+$ and $\{[\mu_n^-, \lambda_n^-] : n \in \mathbb{N}\} \subseteq \mathcal{F}^-$. Let $\mathcal{A}$ be the filter generated by the collection of sets $\{uv : \lambda_n^+ \leq u \leq \mu_n^+, \hat{\lambda}_n^+ \leq v \leq \hat{\mu}_n^+\}, n \in \mathbb{N}$. Since $\lambda_n^+, \hat{\lambda}_n^+ \geq 0$ for all $n \in \mathbb{N}$, it follows from (10) that the inequalities $\lambda_n^+\hat{\lambda}_n^+ \leq u\hat{\lambda}_n^+ \leq uv \leq u\hat{\mu}_n^+ \leq \mu_n^+\hat{\mu}_n^+$ hold for all $u \in [\lambda_n^+, \mu_n^+]$ and $v \in [\hat{\lambda}_n^+, \hat{\mu}_n^+]$. Therefore $\{uv : \lambda_n^+ \leq u \leq \mu_n^+, \hat{\lambda}_n^+ \leq v \leq \hat{\mu}_n^+\} \subseteq [\lambda_n^+\hat{\lambda}_n^+, \mu_n^+\hat{\mu}_n^+]$ for each $n \in \mathbb{N}$ so that $[\{[\lambda_n^+\hat{\lambda}_n^+, \mu_n^+\hat{\mu}_n^+] : n \in \mathbb{N}\}] \subseteq \mathcal{A}$. The sequence $(\lambda_n^+\hat{\lambda}_n^+)$ is increasing, while the sequence $(\mu_n^+\hat{\mu}_n^+)$ is decreasing. Since (λ_n^+) increases to f^+ and $(\hat{\lambda}_n^+)$ increases to g^+ it follows from (12) that

$$\sup\{\lambda_m^+\hat{\lambda}_n^+ : m, n \in \mathbb{N}\} = \sup_{m\in\mathbb{N}} \lambda_m^+ \cdot \sup_{n\in\mathbb{N}} \hat{\lambda}_n^+ = f^+g^+. \tag{17}$$

Since both sequences (λ_n^+) and $(\hat{\lambda}_n^+)$ are increasing and positive, it follows from (11) that $\lambda_m^+\hat{\lambda}_n^+ \leq \lambda_k^+\hat{\lambda}_k^+$ for all $k, m, n \in \mathbb{N},\ m, n \leq k$. These inequalities, together with the identity (17), imply that $\sup_{n\in\mathbb{N}} \lambda_n^+\hat{\lambda}_n^+ = f^+g^+$ so that the sequence $(\lambda_n^+\hat{\lambda}_n^+)$ increases to f^+g^+. In the same way, it follows that the sequence $(\mu_n^+\hat{\mu}_n^+)$ decreases to f^+g^+. Since $[\lambda_n^+\hat{\lambda}_n^+, \mu_n^+\hat{\mu}_n^+] \in \mathcal{A}$ for all $n \in \mathbb{N}$, it follows that $\mathcal{A}$ converges to f^+g^+. Since $[\lambda_n^+, \mu_n^+] \in \mathcal{F}^+$ and $[\lambda_n^-, \mu_n^-] \in \mathcal{F}^-$ it follows that $\mathcal{A} \subseteq \mathcal{F}^+ \cdot \mathcal{G}^+$, thus $\mathcal{F}^+ \cdot \mathcal{G}^+ \in \lambda_o(f^+g^+)$. In the same way it follows that $\mathcal{F}^- \cdot \mathcal{G}^- \in \lambda_o(f^-g^-)$, $\mathcal{F}^+ \cdot \mathcal{G}^- \in \lambda_o(f^+g^-)$ and $\mathcal{F}^- \cdot \mathcal{G}^+ \in \lambda_o(f^-g^+)$. Since λ_o is a vector space convergence structure, see Theorem 14, it therefore follows that $\mathcal{F}^+ \cdot \mathcal{G}^+ + \mathcal{F}^- \cdot \mathcal{G}^- - \mathcal{F}^+ \cdot \mathcal{G}^- - \mathcal{F}^- \cdot \mathcal{G}^+$ converges to $f^+g^+ + f^-g^- - f^+g^- - f^-g^+$ with respect to the order convergence structure. The identity (13) and the inclusion (16) now imply that $\mathcal{F} \cdot \mathcal{G}$ converges to fg. Thus λ_o is an algebra convergence structure on F.

Conversely, assume that λ_o is an algebra convergence structure on F. Let (u_n) and (v_n) be sequence in F^+ that increase to u and v, respectively. Then $(u_nv_n) \subseteq \{u_nv_m : m, n \in \mathbb{N}\}$ is an increasing sequence by (11), and since λ_o is an algebra convergence structure, (u_nv_n) converges to uv. Hence $\sup\{u_nv_m : m, n \in \mathbb{N}\} = \sup_{n\in\mathbb{N}} u_nv_n = uv$. □

4.2. *Continuous linear mappings*

In this section we consider linear operators $T : E \to F$ acting between Archimedean Riesz spaces E and F. In particular, we characterize the bounded and the continuous operators with respect to the order convergence structure. We first consider the particular case when the operator T is positive. This result is then used to characterize regular, continuous operators. As is customary in the literature, we denote the space of σ-order continuous operators by $L_c(E, F)$. We write $E_c^\sim$ for the σ-order continuous linear functions on E.

Proposition 17. *Let E and F be Archimedean Riesz spaces. A positive linear mapping $T : E \to F$ is continuous with respect to the order convergence structure if and only if T is σ-order continuous.*

Proof. Let T be σ-order continuous and let $\mathcal{F}$ converge to 0. Let (λ_n) and (μ_n) be the sequences associated with $\mathcal{F}$ through (5). Since $T \geq 0$ is σ-order continuous, the sequence $(T\lambda_n)$ increases to 0 and $(T\mu_n)$ decreases to 0. Since T is positive, $T[\lambda_n, \mu_n] \subseteq [T\lambda_n, T\mu_n]$ for all $n \in \mathbb{N}$. Hence $[\{[T\lambda_n, T\mu_n] : n \in \mathbb{N}\}] \subseteq T(\mathcal{F})$ so that $T(\mathcal{F})$ converges to 0 in F. Therefore T is continuous at 0 and thus continuous on E. The converse is trivial. □

Theorem 18. *Let E and F be Archimedean Riesz spaces. If a linear mapping $T : E \to F$ is continuous with respect to the order convergence structure, then T is σ-order continuous. If T is a regular σ-order continuous operator, then T is continuous with respect to λ_o.*

Proof. Assume that $T : E \to F$ is continuous, and let (u_n) decrease to 0 in E. Then (u_n) order converges to 0, and hence by Theorem 6 (u_n) converges to 0 with respect to the order convergence structure on E. Since T is continuous, the sequence (Tu_n) converges to 0 in F. Thus Theorem 6 implies that (Tu_n) order converges to 0 in F. Hence it follows from (2) that there is a sequence (λ_n) that decreases to 0 such that $|Tu_n| \leq \lambda_n,\ n \in \mathbb{N}$. Since (λ_n) decreases to 0, it follows that $\inf_{n\in\mathbb{N}} |Tu_n| = 0$. Because T is continuous, it is also bounded, so that T maps bounded sets, with respect to λ_o, in E into bounded sets with respect to λ_o in F. Hence T is order bounded by [11, Proposition 2.1], which shows that T is σ-order continuous.

Now assume that T is a regular σ-order continuous operator. That is, we can express T as $T = S_0 - S_1$ where S_0 and S_1 are positive, σ-order continuous operators. Proposition 17 implies that S_0 and S_1 are continuous with respect to the order convergence structure, so that the result follows from the stated decomposition of T. □

Corollary 19. *Let E and F be an Archimedean Riesz spaces, with F Dedekind complete. A linear operator $T : E \to F$ is continuous with respect to the order convergence structure if and only if T is σ-order continuous.*

In particular, a linear functional $\varphi : E \to \mathbb{R}$ is continuous with respect to the order convergence structure if and only if φ is σ-order continuous.

Following [4], for Archimedean Riesz spaces E and F equipped with the order convergence structure, we denote by $\mathcal{L}(E, F)$ the continuous linear mappings from E into F. In particular, $\mathcal{L}E$ denotes the linear space of continuous linear functionals on E. Thus Theorem 18 states that, for Archimedean Riesz spaces E and F, the inclusion

$$\mathcal{L}(E, F) \subseteq L_c(E, F) \tag{18}$$

holds, and equality in (18) holds whenever F is Dedekind complete. In particular, $\mathcal{L}E = E_c^{\sim}$ for any Archimedean Riesz space E. Thus the dual $\mathcal{L}E$ of E, with respect to the order convergence structure, may be trivial. Indeed, there exists an Archimedean Riesz space E with the property that no linear mapping $\varphi : E \to \mathbb{R}$ is σ-order continuous; see for instance [16, Example 21.6(ii)].

We may note that, in view of (18), the space $\mathcal{L}(E, F)$ is a Dedekind complete, and thus Archimedean, Riesz space whenever F is Dedekind complete. In particular, $\mathcal{L}E$ is Dedekind complete Archimedean Riesz space. In this case, one may consider the order convergence structure on $\mathcal{L}E$. With the convergence structure λ_o, the space $\mathcal{L}E$ is a Hausdorff, regular and first countable convergence vector space. This insight will be used in [12] to construct a topological duality theory for a class of Archimedean Riesz spaces.

4.3. Completeness and completion

As the title of this section indicates, we now proceed to consider issues of completeness and completion of the order convergence structure on an Archimedean Riesz space. As it turns out, and as expected, completeness with respect to the convergence structure λ_o is equivalent to a form of completeness with respect to the order on the Riesz space. In particular, we show that F is complete with respect to the order convergence structure if and only if it is order complete. We further construct the convergence vector space completion of F as a certain order complete Riesz space, equipped with a suitable vector space convergence structure. The way in which this convergence structure relates to the order convergence structure is also discussed.

As a first step in addressing the issue of completeness, we characterize the Cauchy sequences with respect to the order convergence structure. In this regard, recall [15, Exercise 101.8] that a sequence (u_n) in a Riesz space F is *order Cauchy* whenever there is a sequence $(\mu_n) \subset F$ that decreases to 0 such that

$$-\mu_n \leq u_n - u_m \leq \mu_n, \quad m, n \in \mathbb{N},\ m \geq n. \tag{19}$$

F is called *order complete* if every order Cauchy sequence in F order converges to some $u \in F$.

A sequence (u_n) in F is a Cauchy sequence with respect to the order convergence structure if and only if the filter $\langle u_n \rangle - \langle u_n \rangle$ converges to 0. Since λ_o is first countable, it follows from [4, Proposition 3.6.5] that F is complete with respect to the order convergence structure if and only very Cauchy sequence converges to some $u \in E$.

Proposition 20. *Let F be an Archimedean Riesz space. A sequence (u_n) on F is a Cauchy sequence with respect to the order convergence structure if and only if (u_n) is order Cauchy.*

Proof. Suppose that $(u_n) \subset F$ is an order Cauchy sequence. Let $\langle u_n \rangle$ denote the Frechét filter generated by (u_n). The filter $\langle u_n \rangle - \langle u_n \rangle$ is generated by the collection of sets $G_N = \{u_n - u_m : n, m \geq N\}$, $N \in \mathbb{N}$. Let (μ_n) be the sequence associated with (u_n) through (19). Then it follows by (19) that $G_n \subseteq [-\mu_n, \mu_n]$, $n \in \mathbb{N}$. Thus the inclusion $\mathcal{G} = [\{[-\mu_n, \mu_n] : n \in \mathbb{N}\}] \subseteq \langle u_n \rangle - \langle u_n \rangle$ holds. Since (μ_n) decreases to 0, the sequence $(-\mu_n)$ increases to 0. Consequently the filter $\mathcal{G}$ converges to 0. Therefore $\langle u_n \rangle - \langle u_n \rangle$ converges to 0 so that (u_n) is a Cauchy sequence with respect to λ_o.

Conversely, suppose that the sequence (u_n) is a Cauchy sequence with respect to the order convergence structure. That is, the filter $\langle u_n \rangle - \langle u_n \rangle$, with basis $\{G_N : N \in \mathbb{N}\}$ belongs to $\lambda_o(0)$. According to (5) it therefore follows that there are sequences $(\lambda_n), (\mu_n) \subset F$ that increases to 0 and decreases to 0, respectively, such that $\{[\lambda_n, \mu_n] : n \in \mathbb{N}\} \subseteq \langle u_n \rangle - \langle u_n \rangle$. Thus for each $k \in \mathbb{N}$ there exists $N(k) \in \mathbb{N}$ such that $G_{N(k)} \subseteq [\lambda_k, \mu_k]$. Define the sequence $(\hat{\mu}_n)$ by $\hat{\mu}_n = \mu_n \vee (-\lambda_n), n \in \mathbb{N}$. Note that $-\hat{\mu}_n = (-\mu_n) \wedge \lambda_n$ for all $n \in \mathbb{N}$, according to [7, Theorem 11.5(vi)]. Since we clearly have $-\hat{\mu}_n \leq \lambda_n \leq \mu_n \leq \hat{\mu}_n$ for each $n \in \mathbb{N}$, it follows from $G_{N(k)} \subseteq [\lambda_k, \mu_k]$ that for each $k \in \mathbb{N}$ there exists $N(k) \in \mathbb{N}$ such that $-\hat{\mu}_k \leq u_n - u_m \leq \hat{\mu}_k$ for all $m, n \in \mathbb{N}$, $m, n \geq N(k)$. Since (λ_n) increases to 0, the sequence $(-\lambda_n)$ decreases to 0. Therefore, see [7, Theorems 13.1(i) and 15.3], it follows that the sequence $(\hat{\mu}_n)$ decreases to 0. Now define the sequence $(\bar{\mu}_n)$ as $\bar{\mu}_n = \inf\left\{\hat{\mu}_k : \hat{\mu}_k \leq \mu_l - \mu_m \leq \hat{\mu}_k,\ m, l \geq n\right\}$. Since $-\hat{\mu}_k \leq u_n - u_m \leq \hat{\mu}_k$ and $(\hat{\mu}_n)$ decreases to 0, the sequence $(\bar{\mu}_n)$ is well defined. Furthermore, $(\bar{\mu}_n)$ is clearly a decreasing sequence, and for all $k \in \mathbb{N}$ there exists $N(k) \in \mathbb{N}$ such that $0 \leq \bar{\mu}_n \leq \hat{\mu}_k$ for all $n \in \mathbb{N}$, $n \geq N(k)$. Since $(\hat{\mu}_n)$ decreases to 0, it now follows that $(\bar{\mu}_n)$ decreases to 0 as well. According to the definition of $(\bar{\mu}_n)$, the sequences $(\bar{\mu}_n)$ and (u_n) satisfy the inequalities $-\bar{\mu}_n \leq u_n - u_m \leq \bar{\mu}_n$ for all $n \in \mathbb{N}$ and every $m \geq n$. Therefore (u_n) is an order Cauchy sequence. □

Corollary 21. *Let F be an Archimedean Riesz space. Then F is complete with respect to the order convergence structure if and only if F is order complete.*

Proof. Since λ_o is first countable by Corollary 9, F is a complete convergence vector space if and only if it is sequentially complete [4, Proposition 3.6.5]. By Proposition 20 and Corollary 12, it now follows that F is a complete convergence vector space if and only if F is order complete. □

Recall [7] that a Riesz space F is order separable whenever every set $A \subset F$ with $u_0 = \sup A \in F$ has countable subset C such that $u_0 = \sup C$.

Theorem 22. *Let F be an Archimedean Riesz space. If F is Dedekind σ-complete, then F is complete with respect to the order convergence structure. If F is order separable, the converse is also true.*

Proof. Assume that F is Dedekind σ-complete. Since any Dedekind σ-complete Riesz space is order complete [15, page 696], it follows by Corollary 21 that F is a complete convergence vector space with respect to λ_o.

Now assume that F is order separable, and that F is a complete convergence vector space with respect to λ_o. It is sufficient to prove that any increasing sequence $(u_n) \subset F^+$ bounded from above by some $u \in F^+$ has supremum in F. Denote by V the set of all upper bounds of (u_n). Since F is Archimedean, [7, Theorem 22.5] implies that

$$\inf\{v - u_n : n \in \mathbb{N}, v \in V\} = 0. \tag{20}$$

Since F is order separable, it follows that there exists a countable subset $\{v_k : k \in \mathbb{N}\}$ of V such that $\inf\{v_k - u_n : k, n \in \mathbb{N}\} = 0$. Define the sequence (w_n) in F as $w_n = \inf\{v_k : k \leq n\}$. The sequence w_n is clearly decreasing, and in view of (20) it satisfies the inequalities $u_n \leq w_n,\ n \in \mathbb{N}$ and the identity $\inf\{w_n - u_n : n \in \mathbb{N}\} = 0$. Since the sequence (u_n) is increasing, it follows by [9, Proposition 2.7] that (u_n) is order Cauchy. Furthermore, F is a complete convergence vector space with respect to λ_o so that Corollary 12 and Proposition 20 imply that (u_n) order converges to some $u \in F$, and since (u_n) is increasing it follows by [7, Theorem 16.1] that $u = \sup_{n \in \mathbb{N}} u_n$. Therefore F is Dedekind σ-complete. □

We now turn to the issue of constructing the convergence vector space completion of an Archimedean Riesz space with respect to the order convergence structure. In this regard, let $\overline{F}$ denote the Dedekind completion of F, and consider F to be a subset of $\overline{F}$. Define $\hat{F}$ as

$$\hat{F} = \left\{ u \in \overline{F} \;\middle|\; \begin{array}{ll} \exists & (\lambda_n), (\mu_n) \subset F : \\ (1) & \lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n,\ n \in \mathbb{N} \\ (2) & \sup\limits_{n \in \mathbb{N}} \lambda_n = u = \inf\limits_{n \in \mathbb{N}} \mu_n \end{array} \right\}. \tag{21}$$

Remark 23. Recall [1, Chap. 1, Definition 2.1] that a Riesz space F is almost Dedekind σ-complete if it is Riesz isomorphic to a super order dense subspace of a Dedekind σ-complete Riesz space. For such a Riesz space, the Dedekind σ-completion of F, in the sense of Quin [9], is precisely the set $\hat{F}$; see [1, page 11].

Proposition 24. *For any Archimedean Riesz space F, the set $\hat{F}$ is an order complete Riesz subspace of $\overline{F}$ that contains F.*

Proof. That $\hat{F}$ is a Riesz subspace of $\overline{F}$ follows by [7, Theorems 13.1, 15.2 and 15.3]. Furthermore, the inclusion $F \subseteq \hat{F}$ is obvious.

Now we prove that $\hat{F}$ is order complete. In this regard, let (u_n) be an order Cauchy sequence on $\hat{F}$. According to [9, Proposition 2.7] it follows that there exists an increasing sequence (λ_n) and a decreasing sequence $(\mu_n) \subset$ in $\hat{F}$ such that $\inf\{\mu_n - \lambda_n : n \in \mathbb{N}\} = 0$ and $\lambda_n \leq u_n \leq \mu_n$, $n \in \mathbb{N}$. Since (λ_n) is bounded from above, and (μ_n) is bounded from below, there exist $u_0, u_1 \in \overline{F}$ such that $\sup_{n\in\mathbb{N}} \lambda_n = u_0$ and $\inf_{n\in\mathbb{N}} \mu_n = u_1$. Since $\inf_{n\in\mathbb{N}}(\mu_n - \lambda_n) = 0$. It follows from [7, Theorems 13.1(ii) and 15.2(iii)] that $u = u_0 = u_1$. Therefore $u \in \hat{F}$ and $\sup_{n\in\mathbb{N}} \lambda_n = u = \inf_{n\in\mathbb{N}} \mu_n$. According to the definition of $\hat{F}$ given in (21), we may find for each $n \in \mathbb{N}$ an increasing sequence $(\lambda_m^{(n)}) \subset F$ such that $\lambda_n = \sup_{m\in\mathbb{N}} \lambda_m^{(n)}$. Then [2, Lemma 36] implies that the sequence $(\hat{\lambda}_n)$ defined through $\hat{\lambda}_n = \sup\{\lambda_n^{(1)}, \ldots, \lambda_n^{(n)}\}$, $n \in \mathbb{N}$ increases to u. In the same way we can construct a sequence $(\hat{\mu}_n)$ in F that decreases to u. Therefore $u \in \hat{F}$. Moreover, since $\sup_{n\in\mathbb{N}} \lambda_n = u = \inf_{n\in\mathbb{N}} \mu_n$ and $\lambda_n \leq u_n \leq \mu_n$ it follows that (u_n) order converges to u. $\square$

Corollary 25. *For any Archimedean Riesz space F, the Riesz space $\hat{F}$ is a complete convergence vector space with respect to the order convergence structure.*

Proposition 26. *Let F be an Archimedean Riesz space, and equip $\hat{F}$ with the order convergence structure. Then the subspace convergence structure induced on F coincides with the order convergence structure on F.*

Proof. Denote by λ_o' the subspace convergence structure induced on F by the order convergence structure on $\hat{F}$. The inclusion $\lambda_o(u) \subseteq \lambda_o'(u)$ obviously holds. Conversely, consider any $u \in F$ and any filter $\mathcal{F} \in \lambda_o'(u)$. Then there exists a filter $\mathcal{G}$ on $\hat{F}$ that converges to u with respect to the order convergence structure on $\hat{F}$ such that $\mathcal{F} = \{G \cap F : G \in \mathcal{G}\}$. Let (λ_n) and (μ_n) denote the sequences in $\hat{F}$ associated with $\mathcal{G}$ through (5). By the definition of $\hat{F}$ given in (21), we may find for each $n \in \mathbb{N}$ an increasing sequence $(\lambda_m^{(n)}) \subset F$ such that $\lambda_n = \sup_{m\in\mathbb{N}} \lambda_m^{(n)}$. Then [2, Lemma 36] implies that the sequence $(\hat{\lambda}_n) \subseteq F$ defined as $\hat{\lambda}_n = \sup\{\lambda_n^{(1)}, \lambda_n^{(2)}, \ldots, \lambda_n^{(n)}\}$ increases to u, while the sequence $(\hat{\mu}_n) \subset F$ defined by $\hat{\mu}_n = \inf\{\mu_n^{(1)}, \mu_n^{(2)}, \ldots, \mu_n^{(n)}\}$ decreases to u. Furthermore, the inequalities $\hat{\lambda}_n \leq \lambda_n \leq u \leq \mu_n \leq \hat{\mu}_n$ hold for all $n \in \mathbb{N}$ so that the inclusion

$$\begin{aligned}\{v \in F : \hat{\lambda}_n \leq v \leq \hat{\mu}_n\} &\supseteq \{v \in F : \lambda_n \leq v \leq \mu_n\} \\ &= [\lambda_n, \mu_n] \cap F\end{aligned}$$

holds for all $n \in \mathbb{N}$. The filter $\mathcal{H} = [\{\{v \in F : \hat{\lambda}_n \leq v \leq \hat{\mu}_n\} : n \in \mathbb{N}\}]$ clearly converges to u. Therefore $\mathcal{H} \subseteq \mathcal{F}$ so that $\mathcal{F} \in \lambda_o(u)$. Thus the inclusion $\lambda_o'(u) \subseteq \lambda_o(u)$ holds. This shows that $\lambda_o(u) = \lambda_o'(u)$, $u \in F$ where λ_o denotes the order convergence structure on F. $\square$

For any Archimedean Riesz space F, it is clear that F is dense in $\hat{F}$ with respect to the order convergence structure; see (4). In particular, for every $u \in \hat{F}$ there is a sequence (u_n) in F which order converges to u in $\hat{F}$. Therefore Proposition 26 may seem to suggest that $\hat{F}$, equipped with the order convergence structure, is the convergence vector space completion of F. However, this is not the case, since the universal property (3) of convergence vector space completions may fail; see Example 33 at the end of this section.

In view of these remarks, a slight modification of the order convergence structure is now introduced: For any Archimedean Riesz space F, and any $u \in \hat{F}$, we denote by $\lambda_o^\sharp(u)$ the collection of filters $\mathcal{F}$ on $\hat{F}$ that satisfy the following: There exist an increasing sequence (λ_n) and a decreasing sequence (μ_n) in $\hat{F}$ such that

$$\sup_{n\in\mathbb{N}} \lambda_n = u = \inf_{n\in\mathbb{N}} \mu_n \quad \text{and} \quad [\{\{v \in F : \lambda_n \leq v \leq \mu_n\} : n \in \mathbb{N}\}] \subseteq \mathcal{F} \tag{22}$$

or $\mathcal{F} = [u]$. Note that, if $\hat{F} \neq F$, then $\lambda_o^\sharp$ is strictly finer than λ_o. The following results follow by the same arguments used in the proofs of Theorems 4 and 14 and Proposition 20.

Theorem 27. *Let F be an Archimedean Riesz space. Then $\lambda_o^\sharp$ is a Hausdorff vector space convergence structure on $\hat{F}$ with the following properties:*

(i) *$\lambda_o^\sharp$ is first countable.*
(ii) *A sequence (u_n) in $\hat{F}$ is a Cauchy sequence with respect to $\lambda_o^\sharp$ if and only if (u_n) is an order Cauchy sequence.*
(iii) *A sequence (u_n) in $\hat{F}$ converges to $u \in \hat{F}$ if and only if (u_n) order converges to u.*

Corollary 28. *For any Archimedean Riesz space F, the convergence structure $\lambda_o^\sharp$ is a complete vector space convergence structure on $\hat{F}$.*

Proof. Since $\hat{F}$ is order complete by Proposition 24, it follows by Theorem 27 that every Cauchy sequence with respect to $\lambda_o^\sharp$ converges with respect to $\lambda_o^\sharp$. Since $\lambda_o^\sharp$ is first countable by Theorem 27(iii), the result follows by [4, Proposition 3.6.5]. □

Theorem 29. *Let F be an Archimedean Riesz space, equipped with the order convergence structure. The convergence vector space completion of F is $\hat{F}$ equipped with the convergence structure $\lambda_o^\sharp$.*

Proof. We first have to show that the subspace convergence structure that F inherits from $\hat{F}$ is the order convergence structure. This follows in the same way as the proof of Proposition 26, and so we omit the details.

Furthermore, F is clearly dense in $\hat{F}$, so it remains to verify the universal property of the convergence vector space completion; see (3). In this regard, let E be a complete, Hausdorff convergence vector space, and $T : F \rightarrow E$ a continuous linear mapping. The extension $\hat{T} : \hat{F} \rightarrow E$ is defined in the usual way. In particular, since T is continuous and linear, it follows that $T(\mathcal{F}) - T(\mathcal{F}) = T(\mathcal{F} - \mathcal{F})$ converges to 0 in E for every Cauchy filter $\mathcal{F}$ on F. Therefore $T(\mathcal{F})$ is a Cauchy filter on E for every Cauchy filter $\mathcal{F}$ on F. In particular, since E is complete, it follows that for each Cauchy filter $\mathcal{F}$ on F, there exists $f \in E$ such that $T(\mathcal{F})$ converges to f in E. Since F is dense in $\hat{F}$, every $u \in \hat{F}$ is the limit of a Cauchy filter on F so that the set $C_u = \{\mathcal{F} \text{ a Cauchy filter on } F : [\mathcal{F}]_{\hat{F}} \in \lambda_o^\sharp(u)\}$ is nonempty. Furthermore, $\mathcal{F} \cap \mathcal{G} \in C_u$ for all $u \in \hat{F}$ and $\mathcal{F}, \mathcal{G} \in C_u$. Define the mapping $\hat{T}$ as $\hat{T} : \hat{F} \ni u \mapsto \lim T(\mathcal{F}) \in E$ where $\mathcal{F} \in C_u$. Clearly $\hat{T}$ defines an extension of T. Since $\mathcal{F} \cap \mathcal{G} \in C_u$ for $\mathcal{F}, \mathcal{G} \in C_u$ and E is Hausdorff it follows that $\hat{T}u$ is independent of the choice of $\mathcal{F} \in C_u$. The linearity of $\hat{T}$ follows by a standard argument [6]. Thus we show only that $\hat{T}$ is continuous. In this regard, we claim that for all $u \in \hat{F}$ and $\mathcal{F} \in \lambda_o^\sharp(u)$ there is a Cauchy filter $\mathcal{G}$ on F such that $[\mathcal{G}]_{\hat{F}} \subseteq \mathcal{F}$. Let $\mathcal{F}$ converge to $u \in \hat{F}$, and let (λ_n) and (μ_n) be the sequences in $\hat{F}$ associated with u through (22). According to the definition (21) of $\hat{F}$ it follows that we may find for each $n \in \mathbb{N}$ an increasing sequence $(\lambda_m^{(n)}) \subset F$ such that $\lambda_n = \sup_{m\in\mathbb{N}} \lambda_m^{(n)}$. We now apply [2, Lemma 36] to the sequences $(\lambda_m^{(n)})$, with $n \in \mathbb{N}$. In particular, since (λ_n) increases to u, it follows that the sequence $(\hat{\lambda}_n)$ defined as $\hat{\lambda}_n = \sup\{\lambda_n^{(1)}, \ldots, \lambda_n^{(n)}\}$, $n \in \mathbb{N}$ increases to u. In the same way, it follows that the sequence $(\hat{\mu}_n) \subset F$ defined as $\hat{\mu}_n = \inf\{\mu_n^{(1)}, \mu_n^{(2)}, \ldots, \mu_n^{(n)}\}$, $n \in \mathbb{N}$ decreases to u. Therefore [7, Theorem 13.1] implies that $\inf_{n\in\mathbb{N}}(\hat{\mu}_n - \hat{\lambda}_n) = 0$. Since $(\hat{\lambda}_n)$ is increasing and $(\hat{\mu}_n)$ is decreasing, it therefore follows by Lemma 30 that the filter $\mathcal{G} = [\{[\hat{\lambda}_n, \hat{\mu}_n] : n \in \mathbb{N}\}]$ is a Cauchy filter on F. Furthermore, since $\lambda_n \leq \hat{\lambda}_n \leq \hat{\mu}_n \leq \mu_n$ it follows that $\mathcal{G} \subseteq \mathcal{F}$, which verifies our claim. Now note that $\hat{T}([\mathcal{G}]_{\hat{F}}) = [\{\hat{\hat{T}}(G) : G \in \mathcal{G}\}] = [\{T(G) : G \in \mathcal{G}\}] = T(\mathcal{G})$. Therefore the inclusion $[\mathcal{G}]_{\hat{F}} \subseteq \mathcal{F}$ implies that $T(\mathcal{G}) \subseteq \hat{T}(\mathcal{F})$. Since $T(\mathcal{G})$ converges to $\hat{T}(u)$ by the definition of $\hat{T}$, it now follows that $\hat{T}(\mathcal{F})$ converges to $\hat{T}(u)$ in E. Therefore $\hat{T}$ is continuous. This completes the proof. □

The proof of Theorem 29 depends on the following.

Lemma 30. *Let F be an Archimedean Riesz space. Suppose that a filter $\mathcal{F}$ on F satisfies the following: There exist an increasing sequence (λ_n) and a decreasing sequence (μ_n) in F*

$$\inf\{\mu_n - \lambda_n : n \in \mathbb{N}\} = 0 \quad \textit{and} \quad [\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}. \tag{23}$$

Then $\mathcal{F}$ is a Cauchy filter with respect to λ_o.

Proof. Suppose that $\mathcal{F}$ is a filter on F that satisfies (23). Note that the filter $\mathcal{F} - \mathcal{F}$ is generated by $\{G - G = \{u - v : u, v \in G\} : G \in \mathcal{F}\}$. Note also that $[\lambda_n - \mu_n, \mu_n - \lambda_n] \subseteq [\lambda_n, \mu_n] - [\lambda_n, \mu_n]$, $n \in \mathbb{N}$, so that $[\{[\lambda_n - \mu_n, \mu_n - \lambda_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F} - \mathcal{F}$. According to our assumption (23), the sequence $(\mu_n - \lambda_n)$ decreases to 0. Therefore the sequence $(\lambda_n - \mu_n)$ increases to 0 so that $\mathcal{F} - \mathcal{F} \in \lambda_o(0)$. Thus the filter $\mathcal{F}$ is a Cauchy filter with respect to the order convergence structure. □

The following is an immediate corollary to Theorem 29.

Corollary 31. *Let E and F be Archimedean Riesz spaces, with F Dedekind complete. If $T : E \to F$ is a σ-order continuous linear mapping, then there exists a unique σ order continuous linear mapping $\hat{T} : \hat{E} \to F$ such that $\hat{T}u = Tu$ for all $u \in E$.*

Proof. It follows directly from Theorems 22, 18 and 29 that for every σ-order continuous linear mapping $T : E \to F$ there exists a unique linear mapping $\hat{T} : \hat{E} \to F$ which is continuous with respect to the convergence structure $\lambda_o^\sharp$ on $\hat{E}$, and extends T. The proof that $\hat{T}$ is σ-order continuous follows by the same arguments used in the proof of Theorem 18. □

Corollary 32. *Let E and F be Archimedean Riesz spaces, and $T : E \to F$ a positive, continuous linear mapping. Then the continuous extension $\hat{T} : \hat{E} \to \hat{F}$ of T is positive as well.*

Example 33. Let E be an Archimedean Riesz space which is not complete with respect to λ_o. Denote by $\hat{E}_0$ the Riesz space $\hat{E}$, equipped with the convergence structure $\lambda_o^\sharp$. Then $\hat{E}_0$ is a complete convergence vector space, and the identity mapping $\mathrm{Id} : E \to \hat{E}_0$ is a continuous linear mapping with respect to λ_o on E. Since $\lambda_o^\sharp(f) \subsetneq \lambda_o(f)$ for all $f \in \hat{E} \setminus E$, it follows that the extension of the mapping Id, which is the identity mapping on $\hat{E}$, is not continuous with respect to the order convergence structure on $\hat{E}$.

4.4. Completions of Riesz algebras

We now consider the particular case of the results presented in Section 4.3 when F is a Riesz algebra. In this regard, we have the following.

Theorem 34. *Let F be an Archimedean Riesz algebra with σ-order continuous multiplication. Then $\hat{F}$ is a Riesz algebra, with σ-order continuous multiplication, and the convergence structure $\lambda_o^\sharp$ is an algebra convergence structure on $\hat{F}$.*

The proof of Theorem 34 is based on the techniques used in the proof of Theorem 16. Therefore the details are not given here.

It should be noted that Theorem 34 has a significant application in nonlinear theories of generalized functions. In particular, in the context of the Order Completion Method for the solution of systems of nonlinear partial differential equations, spaces of generalized functions are constructed as the completion of suitable Riesz algebras, with respect to an appropriate modification of the order convergence structure. Theorem 34 allows this construction to be interpreted in terms of the nonlinear algebraic theory of generalized functions [10,13].

4.5. $\hat{F}$ and the Dedekind σ-completion

Recall [14] that for any Archimedean Riesz space F, there exists a unique (up to Riesz space isomorphism) Dedekind σ-complete Riesz space $F^\sharp$ and an injective σ-homomorphism

$$C_F : F \to F^\sharp \tag{24}$$

with the following universal property: For every Dedekind σ-complete Riesz space E, and each σ-homomorphism $T : F \to E$ there exists a unique σ-homomorphism $T^\sharp : F^\sharp \to E$ such that the diagram

$$\begin{tikzcd} F \arrow[rr, "T"] \arrow[dr, "C_F"'] & & E \\ & F^\sharp \arrow[ur, "\exists ! T^\sharp"'] & \end{tikzcd} \tag{25}$$

commutes. Furthermore, $C_F(F)$ is σ-dense in $F^\sharp$; see [14, Theorem 5.3]. That is, for each $u \in F^\sharp$ there is a sequence $(u_n) \subset F$ such that $(C_F(u_n))$ order converges to u. Recall [7, Definition 18.10] that for Riesz spaces E and F, a σ-homomorphism from E into F is a linear mapping $T : E \to F$ that satisfies the properties

$$\sup T(A) = T a_0, \qquad T(u \vee v) = T(u) \vee T(v) \tag{26}$$

for all countable sets $A \subseteq E$ with $\sup A = a_0$, and all $y, v \in E$.

In this section we compare the convergence vector space completion $\hat{F}$ constructed in Section 4.3 with the Dedekind σ-completion $F^\sharp$. In this regard, we note that a σ-homomorphism T is monotone and σ-order continuous. Therefore the following is an immediate consequence of Proposition 17.

Corollary 35. *Let E and F be Archimedean Riesz spaces. Every Riesz σ-homomorphism $T : E \to F$ is continuous with respect to the order convergence structure.*

For any Archimedean Riesz space E, the mapping (24) is an injective σ-homomorphism. Thus Corollary 35 implies that C_E is continuous with respect to the order convergence structure on E and $E^\sharp$. Since $E^\sharp$ is Dedekind σ-complete, it is complete with respect to the order convergence structure by Theorem 22. Therefore the mapping (24) extends uniquely to a continuous linear mapping $\hat{C}_E : \hat{E} \to E^\sharp$. Moreover, we have the following.

Theorem 36. *Let E be an Archimedean Riesz space. The mapping $\hat{C}_E$ is an injective Riesz σ-homomorphism. If E is order separable, then $\hat{C}_E$ is a surjection.*

Proof. We first show that $\hat{C}_E$ is a σ-homomorphism. In this regard, we note that $\hat{C}_E$ is positive by Corollary 32. Now suppose that (u_n) decreases to 0 in $\hat{E}$. Theorem 27 implies that (u_n) converges to 0 with respect to the convergence structure $\lambda_o^\sharp$ on $\hat{E}$. Since $\hat{C}_E$ is continuous, it follows that $(\hat{C}_E u_n)$ converges to 0 in $E^\sharp$. Therefore Theorem 6 implies that $(\hat{C}_E u_n)$ order converges to 0. But $\hat{C}_E$ is positive by Corollary 32, since C_E is positive. Therefore $(\hat{C}_E u_n)$ is decreasing in $E^\sharp$. Thus it follows by [7, Theorem 16.1] that $(\hat{C}_E u_n)$ decreases to 0 in $E^\sharp$. Therefore it remains to verify that $\hat{C}_E$ is a Riesz homomorphism. Consider any $u, v \in \hat{E}$. Denote by (λ_n) and (λ_n') the increasing sequences in E associated with u and v, respectively, through (21). That is, $u = \sup_{n\in\mathbb{N}} \lambda_n$ and $v = \sup_{n\in\mathbb{N}} \lambda_n'$. Since $\hat{C}_E$ is continuous, it follows by Theorem 27(iii) that the sequence $(\hat{C}_E(\lambda_n))$ converges to $\hat{C}_E(u)$ in $E^\sharp$. In particular, since $\hat{C}_E$ is positive, it follows from Theorem 6 and [7, Theorem 16.1] that $\hat{C}_E(\lambda_n)$ increases to $\hat{C}_E(u)$ in $E^\sharp$. Similarly, the sequence $(\hat{C}_E(\lambda_n'))$ increases to $\hat{C}_E(v)$ in $E^\sharp$.

But by [7, Theorem 15.3], the sequence (λ_n'') in E, defined by $\lambda_n'' = \lambda_n \vee \lambda_n'$ increases to $u \vee v$ in $\hat{E}$. Therefore $(\hat{C}_E(\lambda_n''))$ converges to $\hat{C}_E(u \vee v)$ in $E^\sharp$. But $\hat{C}_E$ is an extension of C_E, which is a Riesz homomorphism. Since $(\lambda_n'') \subset E$, it therefore follows from the definition of (λ_n'') that

$$\hat{C}_E(\lambda_n'') = C_E(\lambda_n \vee \lambda_n') = C_E(\lambda_n) \vee C_E(\lambda_n') = \hat{C}_E(\lambda_n) \vee \hat{C}_E(\lambda_n'). \tag{27}$$

Since $(\hat{C}_E(\lambda_n))$ increases to $\hat{C}_E(u)$, and $\hat{C}_E(\lambda_n')$ increases to $\hat{C}_E(v)$, the identity (27) and [7, Theorem 15.3] imply that $(\hat{C}_E(\lambda_n''))$ increases to $\hat{C}_E(u) \vee \hat{C}_E(v)$. Therefore Theorem 6 implies that $(\hat{C}_E(\lambda_n''))$ converges to $\hat{C}_E(u) \vee \hat{C}_E(v)$ in $E^\sharp$. Since the order convergence structure is Hausdorff by Corollary 11 it follows that $\hat{C}_E(u \vee v) = \hat{C}_E(u) \vee \hat{C}_E(v)$ for all $u, v \in \hat{E}$. Therefore $\hat{C}_E$ is a Riesz σ-homomorphism.

Now we show that $\hat{C}_E$ is injective. In this regard, consider $u, v \in \hat{E}$ such that $\hat{C}_E(u) = \hat{C}_E(v)$. Let (λ_n) be the increasing sequence in E associated with u, and (μ_n) the decreasing sequence in E associated with v, through (21). According to [7, Theorems 13.1 and 15.2], the sequence $(\hat{\mu}_n) = (\mu_n - \lambda_n)$ decreases to $v - u$. Thus, according to [7, Theorem 15.3], the sequence $(\hat{\mu}_n^+)$ decreases to $(v-u)^+$. Since $\hat{C}_E$ is a σ-homomorphism, it follows that the sequence $(\hat{C}_E(\hat{\mu}_n))$ decreases to $\hat{C}_E(u - v) = \hat{C}_E(u) - \hat{C}_E(v) = 0$ in $E^\sharp$, and therefore $(\hat{C}_E(\hat{\mu}_n^+)) = (\hat{C}_E(\hat{\mu}_n)^+)$ decreases to 0 as well. But $(\hat{\mu}_n^+) \subset E$ so that $\hat{C}_E(\hat{\mu}_n^+) = C_E(\hat{\mu}_n^+)$, $n \in \mathbb{N}$. Therefore $(C_E(\hat{\mu}_n^+))$ decreases to 0 in $E^\sharp$. Since C_E is an injective Riesz σ-homomorphism, it follows that $(\hat{\mu}_n^+)$ decreases to 0 in E. Since $0 \leq (v-u)^+ \leq \hat{\mu}_n^+$ for all $n \in \mathbb{N}$, it follows that $(v-u)^+ = 0$, and hence that $v \leq u$. In the same way, it can be shown that $u \leq v$. This shows that $u = v$, so that $\hat{C}_E$ is injective.

Now assume that E is order separable. Then, according to [7, Theorem 32.9], the Dedekind completion $\overline{E}$ of E is order separable, so that the Riesz space $\hat{E} \subseteq \overline{E}$ is also order separable. According to Corollary 25, $\hat{E}$ is a complete convergence vector space with respect to the order convergence structure. Therefore Theorem 22 implies that $\hat{E}$ is Dedekind σ-complete.

The inclusion mapping $I_E : E \to \hat{E}$ is clearly an injective Riesz σ-homomorphism. According to (25), the Riesz σ-homomorphism I_E extends to a Riesz σ-homomorphism $I_E^\sharp : E^\sharp \to \hat{E}$ with the property that $I_E = I_E^\sharp \circ \hat{C}_E$. Applying the method used to prove that $\hat{C}_E$ is injective, it follows that $I_E^\sharp$ is injective. Furthermore, $I_E^\sharp(\hat{C}_E(u)) = u = \hat{C}_E(I_E^\sharp(u))$ for all $u \in E$ so that the mappings $I_E^\sharp \circ \hat{C}_E : \hat{E} \to \hat{E}$ and $\hat{C}_E \circ I_E^\sharp : E^\sharp \to E^\sharp$ agree with the identity mapping on $\hat{E}$ and $E^\sharp$, respectively, restricted to E. Since E is dense in $\hat{E}$ and $E^\sharp$ with respect to the convergence structure (22) and the order convergence structure, respectively, and all mappings involved are continuous, it follows that $I_E^\sharp \circ \hat{C}_E$ is the identity mapping on $\hat{E}$, and $\hat{C}_E \circ I_E^\sharp$ is the identity mapping on $E^\sharp$. Therefore $\hat{C}_E$ is surjective, and $I_E^\sharp$ is its inverse. $\square$

Theorem 36 shows that the completion $\hat{F}$ is generally *smaller* than the Dedekind σ-completion constructed in [14]. However, if the Riesz space E is order separable, then the two completions agree.

5. Conclusion

We have studied order convergence and the order convergence structure on σ-distributive lattices, with particular emphasis on Archimedean Riesz spaces, and Riesz algebras. The two main applications of the results developed in this paper are presented elsewhere. In [13] results relating to convergence on Riesz algebras are applied to nonlinear theories of generalized functions, while topological duality theories for Archimedean Riesz spaces are investigated in [12].

Acknowledgment

The author would like to thank the referee for his useful comments and suggestions.

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